Analysis of Boolean Functions

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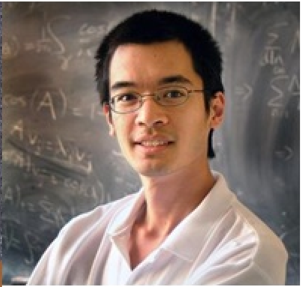
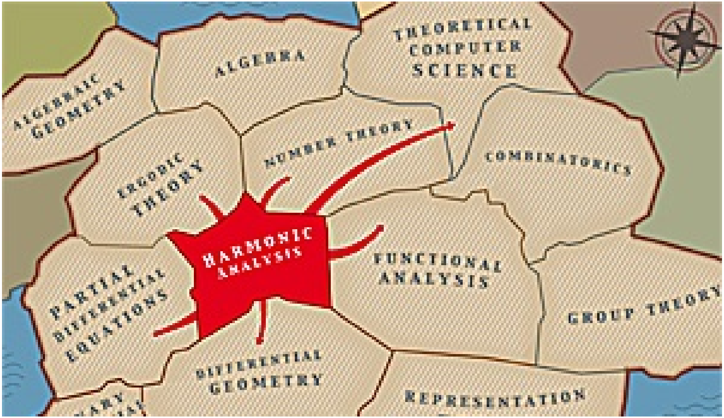
From Chaos to Harmony, Crafoord days , Lund May 2012.

Congratulations Jean and Terry!!!



Harmonic Analysis for Combinatorics and Theoretical Computer Science



Harmonic Analysis for Combinatorics and Theoretical Computer Science - three directions

- I. **Pseudorandomness and structure**

Example: Roth's theorem on 3-terms arithmetic progressions

Tools: Parseval, Hardy-Littlewood circle method, quadratic and higher Fourier analysis,...

Applications: extremal combinatorics, additive number theory and its many own applications, probability theory, theoretical computer science, cryptography

Harmonic Analysis for Combinatorics and Theoretical Computer Science - three directions(cont.)

- II. **Isoperimetry**

Example: Kahn-K.-Linial (1988) (KKL) theorem

Tools: Parseval, Hypercontractivity/log Sobolev

Applications: extremal combinatorics, probability theory and random graphs, computational complexity, and theoretical computer science, game theory

Harmonic Analysis for Combinatorics and Theoretical Computer Science - three directions (cont.)

- III. **Bounds on Error-Correcting Codes**

Example: McEliece, Rodemich, Rumsey, Welch theorem

Tools: Parseval, Delsarte linear programming method, hypergeometric functions.

Applications: Error-correcting codes, sphere packings, packing and covering in combinatorics and geometry

Harmonic Analysis for Combinatorics and Theoretical Computer Science - three directions (cont.)

Problems:

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Harmonic Analysis for Combinatorics and Theoretical Computer Science - three directions (cont.)

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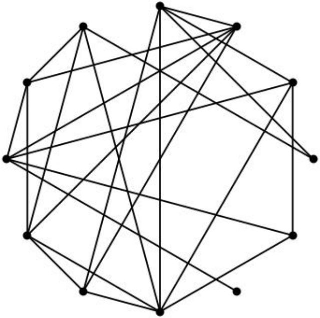
Find other applications of different nature of harmonic analysis to Combinatorics and TCS?

This lecture

This lecture is about the second direction: harmonic analysis applied to discrete isoperimetry. We have several application and potential applications in mind mainly to problems in probability. I will start by mentioning one potential application. It deals with the theory of random graphs initiated by Erdős and Rényi, and the



model $G(n.p)$. In the picture we see a random graph with $n = 12$ and $p = 1/3$.

Threshold and Expectation threshold

Consider a random graph G in $G(n,p)$ and the graph property: G contains a copy of a specific graph H. (Note: H depends on n; a motivating example: H is a Hamiltonian cycle.) Let q be the minimal value for which the expected number of copies of H' in G is at least $1/2$ for every subgraph H' of H. Let p be the value for which the probability that G contains a copy of H is $1/2$.

Conjecture: [Kahn, K. 2006]

$$p/q = O(\log n).$$

The conjecture can be vastly extended to general Boolean functions, and we will hint on possible connection with harmonic analysis and discrete isoperimetry. (Sneak preview: it will require a far-reaching extension of results by Friedgut, Bourgain and Hatami.)

The discrete n-dimensional cube and Boolean functions

The **discrete n-dimensional cube** Ω_n is the set of 0-1 vectors of length n.

A **Boolean function** f is a map from Ω_n to $\{0, 1\}$.

A boolean function f is **monotone** if f cannot decrease when you switch a coordinate from 0 to 1.

The Bernoulli measure

Let $p, 0 < p < 1$, be a real number. The probability measure μ_p is the product probability distribution whose marginals are given by $\mu_p(x_k = 1) = p$. Let $f : \Omega_n \to \{0,1\}$ be a Boolean function.

$$\mu_p(f) = \sum_{x \in \Omega_n} \mu_p(x) f(x) = \mu_p\{x : \ f(x) = 1\}.$$

The total influence

Two vectors in Ω_n are **neighbors** if they differ in one coordinate.

For $x \in \Omega_n$ let $h(x)$ be the number of neighbors y of x such that $f(y) \neq f(x)$.
The **total influence** of f is defined by

$$I^p(f) = \sum_{x \in \Omega_n} \mu_p(x) h(x).$$

If $p = 1/2$ we will omit p as a subscript or superscript.

Russo's lemma

Russo's lemma: For a monotone Boolean function f,

$$d\mu_p(f)/dp = I^p(f).$$

Very useful in percolation theory and other areas.

The **threshold interval** for a monotone Boolean function f is those values of p so that $\mu_p(f)$ is bounded away from 0 and 1. (Say $0.01 \leq \mu_p(f) \leq 0.99$.)

A typical application of Russo's lemma: If for every value p in the threshold interval $I^p(f)$ is large, then the threshold interval itself is short. This is called a **sharp threshold** phenomenon.

(A version of) Harper's theorem

Harper's theorem: If $\mu_p(f) = t$ then

$$I^p(f) \geq 2t \cdot \log_p t.$$

There is a 3 line proof by induction.
Harmonic analysis proof: without the $\log(1/t)$ factor it follows from Parseval.

Influence of variables on Boolean functions

Let

$$\sigma_k(x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n) = (x_1, \ldots, x_{k-1}, 1 - x_k, x_{k+1}, \ldots, x_n).$$

The influence of the kth variable on a Boolean function f is defined by:

$$I_k^p(f) = \mu_p(x \in \Omega_n, f(x) \neq f(\sigma_k(x))).$$

KKL's theorem

Theorem (Kahn, K, Linial, 1988; Bourgain Katznelson KKL 1992; Talagrand 1994 Friedgut K. 1996) There exist a variable k such that

$$I_k(f) \geq C\mu(f)(1-\mu(f)) \log n/n.$$

A sharp version (due to Talagrand)

$$\sum I_k^p(f)/\log(e + I_k^p(f)) \geq C(p)\mu_p(f)(1-\mu_p(f)).$$

Hypercontructivity and Harper's theorem:

We assume now $p = 1/2$. $f = \sum \hat{f}(S) W_S$ is the **Fourier-Walsh** expansion of f. Key ideas:

0 Parseval gives $I(f) = 4 \sum \hat{f}(S)|S|$.

1 Bonami-Gross-Beckner hypercontractive inequality.

$$\| \sum \hat{f}(S)(1/2)^{|S|} \|_2 \le \| f \|_{5/4}.$$

2 For Boolean functions the qth power of the q norm is the measure of the support and does not depend on q. If the support is small this means that the q-norm is very different from the r-norm if $r \neq q$.

(See also : Ledoux' book on concentration of measure phenomena)

Part II: Influence and symmetry

Invariance under transitive group

Theorem: If a monotone Boolean function f with n variables is invariant under a transitive group of permutation of the variables, then

$$I^p(f) \geq C\mu_p(f)(1 - \mu_p(f)) \log n.$$

Proof: Follows from KKL's theorem since all individual influences are the same.

Total influence under symmetry of primitive groups

For a transitive group of permutations $\Gamma \subset S_n$, let $I(\Gamma)$ be the minimum influence for a Γ-invariant function Boolean function with n variables.

Theorem: [Bourgain and K. 1998] If Γ is primitive then one of the following possibilities hold.

- $$I(\Gamma) = \theta(\sqrt{n}),$$
- $$(logn)^{(k+1)/k-o(1)} \leq I(\Gamma) \leq C(\log n)^{(k+1)/k},$$
- $I(\Gamma)$ behaves like $(\log n)\mu(n)$, where $\mu(n) \leq \log \log n$ is growing in an arbitrary way.

Jumps in the behavior of $I(\Gamma)$ for primitive groups Γ

If Γ is not A_n and S_n then $I(\Gamma) \leq (\log n)^2$.

If $I(\Gamma) \leq (\log n)^{1.99}$ then $I(\Gamma) \leq (\log n)^{3/2}$

If $I(\Gamma) \leq (\log n)^{3/2-\epsilon}$ then $I(\Gamma) \leq (\log n)^{4/3}$

If $I(\Gamma) \leq (\log n)^{4/3-\epsilon}$ then $I(\Gamma) \leq (\log n)^{5/4}$

...

If $I(\Gamma) \leq (\log n)^{1+o(1)}$ then $I(\Gamma) \leq \log n \cdot \log \log n$

Threshold behavior for random graphs

The case that Γ is S_n acting on unordered pairs from $[n] = \{1, 2, \ldots, n\}$ describes graph properties. The conclusion is that the threshold interval for graph properties is at most

$$1/\log^{2-o(1)} n.$$

Hypercontractivity and the lower bounds

Both the upper bounds and the lower bounds depend on finding invariants of the group which causes the threshold to go above $\log n$. Giving constructions for the upper bounds requires a detailed understanding of primitive permutation groups based on the classification theorem and O'Nan-Scott theorem.

The lower bounds are based on delicate and complicated harmonic analysis.

Step I: hypercontractivity + random restriction argument + clever inequalities takes you in the graph case from $\log n$ to $\log n^{3/2}$.

Step II: Extremely subtle "bootstrap" to amplify the outcome.

The Entropy Influence conjecture (Friedgut + K. 1996)

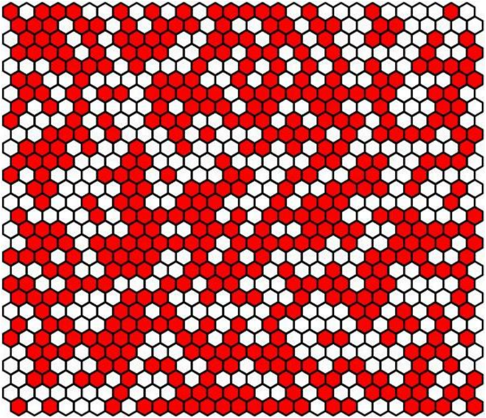
If the Fourier-Walsh expansion of f is $f = \sum \hat{f}(S) W_S$ define

$$E(f) = \sum \hat{f}(S) f^2(S) \log(1/\hat{f}^2(S)).$$

Conjecture: For some absolute constant C,

$$I(f) \geq C \cdot E(f).$$

Scaling-limit symmetry, critical exponents, spectral distribution,...



Prelude: A necessary and sufficient condition for sharp threshold window.

The **Shapley value** of the kth variable is defined by

$$\psi_k(f) = \int_0^1 I_k^p(f)dp.$$

Theorem: (K. 2005)
A necessary and sufficient condition for diminishing threshold window is that the maximum of the Shapley values tends to 0.

Problem: Close the exponential gap in this theorem.

Part III: Stability of Harper's theorem: from Harper to Hatami and beyond

Low Influence and Juntas

A **dictatorship** is a Boolean function depending on one variable. A K-**junta** is a Boolean function depending on K variables.

Theorem: (Friedgut, follows easily from KKL) If p is bounded away from 0 and 1 and $I^p(f) < C$ then f is close to a $K(C)$-Junta.

This works if $\log p / \log n = o(1)$ the most interesting applications would be when p is a power of n. There the theorem is not true.

The works of Friedgut and Bourgain (1999)

Suppose that f is a Boolean function and

$$I^p(f) < pC,$$

then

Friedgut's theorem (1999): If f represent a monotone graph property then f is close to a a "locally defined" function g.

Bourgain's theorem (1999): Unconditionally, f has a substantial "locally defined" ingredient.

Hatami's theorem: Pseudo-juntas

Suppose that for every subset of variables S, we have a function $J_S : \{0,1\}^S -> \{0,1\}$ which can be viewed as a constraint over the variables with indices in S. Now there are two conditions:
A Boolean function is a K-**psudo-junta** if
(1) the expected number of variables in satisfied constraints is bounded by a constant K.
(2) $f(x) = f(y)$ if the variables in satisfied constraints and also their values are the same for x and y.

Hatami's theorem: For every C there is $K(C)$, such that if

$$I^p(f) < pC,$$

then f is close to a $K(C)$-pseudo-junta.

A conjectural extension of Hatami's theorem

Conjecture: Suppose that $\mu_p(f) = t$ and

$$I(f) \leq C \log(1/t)t$$

then f is close to a $O(\log(1/t))$-pseudo-junta.

Stability versions of Harper's theorems

	t bounded away from 0	t small
p bounded away from 0	Friedgut's easy theorem (based on KKL) Many applications to PCP and other areas Some applications to percolation	
p s small	Friedgut's hard THM Bourgain's THM Hatami's THM Many applications for proving sharp threshold behavior: k-SAT, connectivity, Ramsey	Super Hatami Conjecture. Potential applications to finding the location of the threshold, and to other things.

Stability versions for Harper's theorems