Geometry of Poisson brackets and quantum unsharpness

Leonid Polterovich, Tel Aviv University

Jerusalem, December 13, 2012

Mathematical model of classical mechanics

(M^{2n}, ω)–symplectic manifold
ω– **symplectic form**. Locally $\omega = \sum_{i=1}^{n} dp_i \wedge dq_i$.
M-phase space of mechanical system.

Energy determines evolution: $h : M \times [0,1] \to \mathbb{R}$ – Hamiltonian function (energy). Hamiltonian system:

$$\begin{cases} \dot{q} = \frac{\partial h}{\partial p} \\ \dot{p} = -\frac{\partial h}{\partial q} \end{cases}$$

Family of **Hamiltonian diffeomorphisms**

$$\phi_t : M \to M, \;\; (p(0), q(0)) \mapsto (p(t), q(t))$$

Key feature: $\phi_t^* \omega = \omega$.

Symplectic preliminaries

Examples of closed symplectic manifolds:

- Surfaces with an area forms;
- Complex projective manifolds;
- Products.

Poisson bracket: For $f, g \in C^\infty(M)$

$$\{f, g\} = \frac{\partial f}{\partial q} \cdot \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \cdot \frac{\partial g}{\partial q}$$

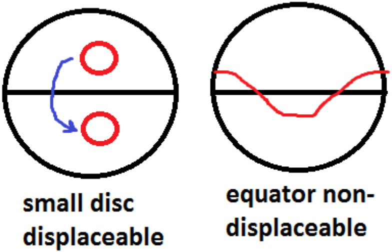
Measures non-commutativity of Hamiltonian flows of f and g.

Small scale on symplectic manifolds

$X \subset M$ **displaceable** if $\exists$ Hamiltonian diffeomorphism ϕ:

$$\phi X \cap X = \emptyset \ .$$

Figure: (Non)-displaceability on the 2-sphere



Non-displaceable fiber theorem

Let $\vec{f} = (f_1, ..., f_N) : M \to \mathbb{R}^N$, $\{f_i, f_j\} = 0$ for all i, j.

Theorem (Entov-P., 2006)

For some $p \in \vec{f}(M) \subset \mathbb{R}^N$, the preimage $\vec{f}^{-1}(p)$ is non-displaceable.

Applications: symplectic topology, integrable systems.
"We will go another way"

$\vec{f} = (f_1, ..., f_N)$– collection of functions.
The magnitude of non-commutativity

$$\nu_c(\vec{f}) = \max_{x,y \in [-1,1]^N} ||\{\sum x_j f_j, \sum y_k f_k\}|| \, .$$

$||f|| := \max |f|$–uniform norm.

The Poisson bracket invariant
$\mathcal{U} = \{U_1, ..., U_N\}$ – a finite open cover of M.
$pb(\mathcal{U}) = \inf \nu_c(\vec{f})$
Infimum over all **partitions of unity** subordinated to $\mathcal{U}$.

Rigidity of partitions of unity

Theorem (Entov-P.-Zapolsky)

If all U_i are displaceable, $pb(\mathcal{U}) > 0$.

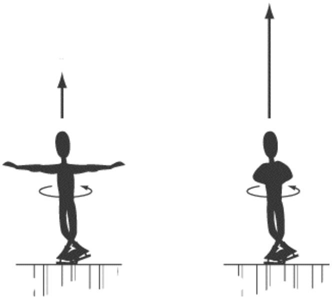
Quantitative version: $pb(\mathcal{U}) \cdot \mathcal{A} \geq C$,
$\mathcal{A}$ – maximal "symplectic size" (e.g. displacement energy) of U_i
(the magnitude of localization)
C depends only on "combinatorics" of the cover $\mathcal{U}$.
Conjecture: C depends only on (M, ω) (universal const??)

INTERPRETATION AND PROOF
RELATED TO QUANTUM MECHANCS

Angular momentum

Phase space – two sphere, $L = (L_1, L_2, L_3) \in S^2$
Attribute of spinning body, depends on angular velocity and shape.

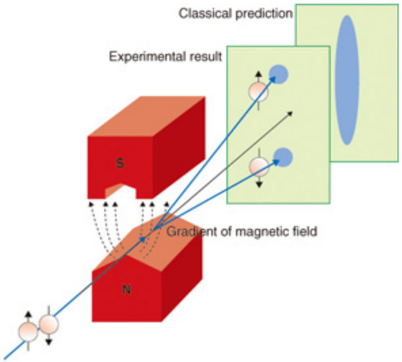
Figure: Conservation of angular momentum:



Poisson bracket relations: $\{L_1, L_2\} = L_3$, ... (cyclic permutation)

Encountering quantum mechanics

Figure: Stern-Gerlach experiment (1922):



Deflection of a beam of atoms of silver through an inhomogeneous magnetic field. L_3 takes two quantized values vs. classical prediction $L_3 \in$ interval.

Naive quantization

H - finite dimensional Hilbert space over $\mathbb{C}$
$\mathcal{L}(H)$ - Hermitian operators on H
$\mathcal{S}$- density operators $\rho \in \mathcal{L}(H)$, $\rho \geq 0$, $Trace(\rho) = 1$.
$\hbar$-Planck constant.
Quantum mechanics contains the classical one in the limit $\hbar \to 0$.

Table: Quantum-Classical Correspondence

	CLASSICAL	QUANTUM
	Symplectic mfd (M, ω)	$\mathbb{C}$-Hilbert space H
OBSERVABLES	$f \in C^\infty(M)$	$A \in \mathcal{L}(H)$
STATES	Probability measures on M	Density ops $\rho \in \mathcal{S}$
BRACKET	Poisson bracket $\{f, g\}$	Commutator $\frac{i}{\hbar}[A, B]$

Mathematical model of quantum mechanics

von Neumann, 1932

$A \in \mathcal{L}(H)$ - observable, $A = \sum \lambda_j P_j$ - spectral decomposition.
In state $\rho \in \mathcal{S}$, A attains value λ_j with probability $Trace(P_j \rho)$.

Example (quantized angular momentum) : $H = \mathbb{C}^2$,
$\widehat{L_1}, \widehat{L_2}, \widehat{L_3}$-Pauli matrices

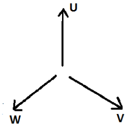
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Commutator relations: $[\widehat{L_1}, \widehat{L_2}] = 2i\widehat{L_3}$, ...

Possible values: $+1; -1$ – explains appearance of only two dots in Stern-Gerlach experiment

A "paradox"

Figure: Back to Stern-Gerlach



Measure projections of the angular momentum L to vectors U, V, W with $U + V + W = 0$. The sum of the results is 0. But possible values are ± 1, contradiction! Deep foundational problem.
An explanation: Gedankenexperiment is impossible: we cannot measure simultaneously non-commuting observables $\widehat{L_U}, \widehat{L_V}, \widehat{L_W}$.
Uncertainty principle:
$Variance(A, \rho) \times Variance(B, \rho) \geq \frac{1}{4} \cdot |Trace([A, B] \cdot \rho)|^2$.

Positive Operator Valued Measures (POVMs)

$\Omega_N = \{1, ..., N\}$, H-Hilbert space, $\mathcal{L}(H)$-Hermitian operators on H.
POVM $A = \{A_1, ..., A_N\}$, $A_j \in \mathcal{L}(H)$, $A_j \geq 0$, $\sum A_j = \mathbb{1}$.
Generalized observable with values in Ω_N.

Statistical axiom: Given the system in the state $\rho \in \mathcal{S}$, the probability of finding A in $j \in \Omega_N$ equals $Trace(A_j \rho)$.

Example: $B \in \mathcal{L}(H)$ – von Neumann observable, $B = \sum_{j=1}^{N} \lambda_j P_j$ – spectral decomposition. Described as **projector valued** POVM $\{P_1, ..., P_N\}$ on $\Omega_N := \{1, ..., N\}$ together with **random variable** $\lambda : \Omega_N \to \mathbb{R}$.
Agrees with von Neumann axiom: In state $\rho \in \mathcal{S}$, B attains value λ_j with probability $Trace(P_j \rho)$.

Quantum noise

$A = \{A_1, ..., A_N\}$ – $\mathcal{L}(H)$-valued POVM on $\Omega_N = \{1, ..., N\}$
$x = (x_1, ..., x_N) : \Omega_N \to \mathbb{R}$ - random variable
$A(x) = \sum x_j A_j$ – operator-valued expectation
Systematic noise $\mathcal{N}(A)$ of A (Ozawa, Busch-Heinonen-Lahti)– certain "magnitude" of the operator valued variance

$$\sum_{j=1}^{N} x_j^2 A_j - A(x)^2$$

Magnitude of non-commutativity:
$\nu_q(A) := \max_{x,y \in [-1,1]^N} |||[A(x), A(y)]|||_{op}$

Theorem (Unsharpness principle)

$\mathcal{N}(A) \geq \frac{1}{2}\nu_q(A)$.

Janssens, 2006; Miyadera-Imai, 2008; P., 2012

Berezin-Toeplitz quantization

(M,ω)-closed symplectic manifold, $[\omega] \in H^2(M,\mathbb{Z})$.
BT-quantization: Sequence of $\mathbb{C}$-Hilbert spaces H_m, $\dim H_m \to \infty$ and linear maps $T_m : C^\infty(M) \to \mathcal{L}(H_m)$, $f \mapsto T_m(f)$:

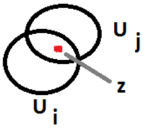
- **(normalization)** $T_m(1) = \mathbb{1}$;
- **(positivity)** $f \geq 0 \Rightarrow T_m(f) \geq 0$.
- **(correspondence principle)**

$$|| [T_m(f), T_m(g)] ||_{op} = \frac{||\{f,g\}||}{m} + O(1/m^2) \, .$$

$\hbar = \frac{1}{m}$–Planck constant, $m \to \infty$– classical limit.
Berezin, 1975; ... Bordeman-Meinrenken-Schlichenmaier, 1994; Guillemin, 1995; Borthwick-Uribe, 1996; Ma-Marinescu, 2008

Classical registration procedure

Figure: Registration:

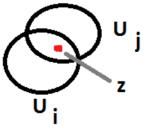


(M,ω) closed symplectic manifold, $\mathcal{U} = \{U_1, ..., U_N\}$-finite open cover, $\vec{f} = \{f_1, ..., f_N\}$-subordinated partition of unity: support$(f_j) \subset U_j$, $f_j \geq 0$, $\sum f_j = 1$.
Each point $z \in M$ has to register in exactly one $U_j \ni z$.

Ambiguity because of overlaps.

Figure: Cell phone registers at an access point :

Figure: Registration:



(M, ω) closed symplectic manifold, $\mathcal{U} = \{U_1, ..., U_N\}$-finite open cover, $\vec{f} = \{f_1, ..., f_N\}$-subordinated partition of unity: support$(f_j) \subset U_j$, $f_j \geq 0$, $\sum f_j = 1$.
Each point $z \in M$ has to register in exactly one $U_j \ni z$.

Ambiguity because of overlaps. Resolve **at random**:
z registers in U_j with probability $f_j(z)$.
"truth, but not the whole truth"

Quantum registration procedure

Observation: T_m – BT-quantization $\Rightarrow$
$A^{(m)} := \{T_m(f_j)\}$ – $\mathcal{L}(H_m)$-valued POVM on $\Omega_N = \{1, ..., N\}$.

Interpretation: Given the system in state $\rho \in \mathcal{S}(H_m)$, probability of registration in U_j equals $Trace(T_m(f_j) \cdot \rho)$.

Theorem

Assume that all U_i's are displaceable. Then

$$\mathcal{N}(A^{(m)}) \geq C(U) \cdot \hbar$$

for all sufficiently small $\hbar = 1/m$.

Ingredients of the proof:

- rigidity of partitions of unity (function theory on symplectic manifolds);
- norm-sensitive correspondence principle (pseudo-differential calculus);
- unsharpness principle for POVMs (linear algebra).

Upgrades under extra assumptions to

Noise-Localization Uncertainty Relation

$$\text{Noise} \times \max_i \text{Size}(U_i) \geq C\hbar \ .$$

EXTRA: Smearing

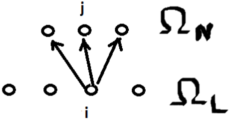


Figure: Smearing of POVMs:

Each state $i \in \Omega_L$ diffuses to $j \in \Omega_N$ with probability γ_{ij}. Statistical procedure given by POVM B on Ω_L transforms to the one given by POVM A on Ω_N with $A_j = \sum_i \gamma_{ij} B_i$.

$\Gamma = (\gamma_{ij}) : \mathbb{R}^N \to \mathbb{R}^L$ – **Markov operator**. $B \mapsto_\Gamma A$

Role of smearing: Two pairwise non-commuting projector valued measures cannot be measured simultaneously. But they can be measured after a smearing–**"unsharp approximate measurements"**.

EXTRA: Noise operator

$A = \{A_1, ..., A_N\}$ – $\mathcal{L}(H)$-valued POVM on $\Omega_N = \{1, ..., N\}$

$x = (x_1, ..., x_N) : \Omega_N \to [-1, 1]^N$ - random variable

$A(x) = \sum x_j A_j$ – operator-valued expectation

$\Delta_A(x) := \sum_{j=1}^N x_j^2 A_j - A(x)^2$ – operator valued variance or **(noise operator)** (Ozawa, Busch-Heinonen-Lahti, 2004)

Difference of variances for POVM- and von Neumann observables
$Trace(\Delta_A(x) \cdot \rho) = \mathbf{Var}(A, \rho) - \mathbf{Var}(A(x), \rho), \forall \rho \in \mathcal{S}$.

EXTRA: Noise and smearing

Let $B \mapsto_\Gamma A$ (B-smearing of A with Markov operator Γ)

Expectations coincide: $\mathbf{Exp}(B, \Gamma x) = \mathbf{Exp}(A, x)\ \forall x \in \mathbb{R}^N$

But noise decreases: $\Delta_B(\Gamma x) \leq \Delta_A(x)$ (Martens-de Muynck)

Can it decrease to zero??

Inherent noise of A: $\mathcal{N}(A) := \inf_{B,\Gamma} \max_{x \in [-1,1]^N} ||\Delta_B(\Gamma x)||_{op}$

Measures "the noise component" persisting under unsmearings

Back to von Neumann axioms (1932)

Quantum state: $\rho : \mathcal{L}(H) \to \mathbb{R}$,

- **(normalization)** $\rho(\mathbb{1}) = 1$;
- **(positivity)** $A \geq 0 \Rightarrow \rho(A) \geq 0$;
- **(linearity)** ρ – linear.

Corollary: $\rho(A) := Trace(A\rho)$, where $\rho \in \mathcal{S}$-density operator.

Corollary: No dispersion free states ("no hidden variables")
$\forall \rho \ \exists A \ : \ Variance(A, \rho) > 0$

Criticism: (Grete Hermann, 1936; Bohm, Bell 1950ies-1960ies):
$\rho(A + B) = \rho(A) + \rho(B)$ makes no sense if A, B are not simultaneously measurable, i.e. $[A, B] \neq 0$.

Quasi-states

Normalized, positive, **quasi-linear** functionals $\zeta : \mathcal{L}(H) \to \mathbb{R}$:

$$\zeta(uA+vB) = u\zeta(A)+v\zeta(B) \; \forall A, B \in \mathcal{L}(H) \; : \; [A, B] = 0, \; \forall u, v \in \mathbb{R} \; .$$

Theorem (Gleason, 1957)

$\dim H \geq 3 \Rightarrow$ *every quasi-state is linear, i.e. a state.*

Apply correspondence principle to define classical analogue of quasi-states: (M, ω)- closed symplectic manifold, $\zeta : C(M) \to \mathbb{R}$-a functional:

- **(normalization)** $\zeta(1) = 1$;
- **(positivity)** $f \geq 0 \Rightarrow \zeta(f) \geq 0$;
- **(quasi-linearity)** $\zeta(uf + vg) = u\zeta(f) + v\zeta(g) \forall f, g \in C(M)$ such that $\{f, g\} = 0$.

Existence of non-linear symplectic quasi-states, 2006-2011

$\dim M = 2$ Aarnes theory of topological quasi-states (1991)

$\dim M \geq 4$ Floer theory–Morse theory for the action functional $\int pdq - hdt$ on loop space of M.
(complex projective space, toric manifolds, blow ups...)
Entov-P., Ostrover, McDuff, Usher, Fukaya-Oh-Ohta-Ono
Extra feature:

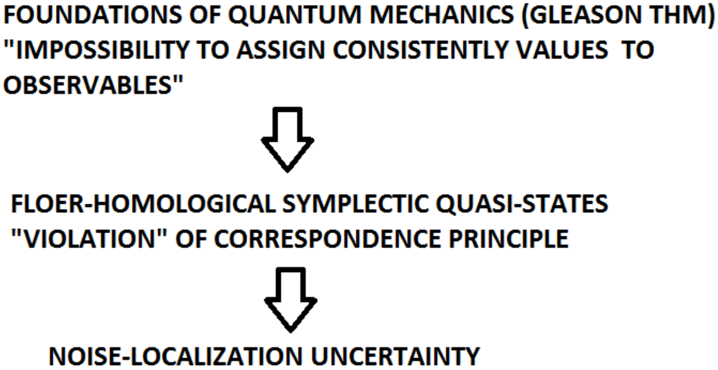
Vanishing: $\zeta(f) = 0$ if support(f) displaceable.

Rigidity of partitions of unity: $\sum f_i = 1$, support(f_i) displaceable. By vanishing, $\zeta(f_i) = 0$. If $\{f_i, f_j\} = 0$,

$1 = \zeta(1) = \zeta(\sum f_i) = \sum \zeta(f_i) = 0$, contradiction. □

Conclusion

Figure: ROAD MAP (physical perspective)



The End

THANK YOU!