GAFA, Geom. funct. anal.
Special Volume – GAFA2000, 978 – 983
1016-443X/00/S20978-6 $ 1.50+0.20/0

GAFA Geometric And Functional Analysis

REFLECTIONS ON THE DEVELOPMENT OF MATHEMATICS IN THE 20TH CENTURY

D. Kazhdan

Let us compare mathematics as it was at the beginning of the century with contemporary mathematics. What has happened in the past hundred years?

To be sure, we have numerous new results and any attempt to list the most important results is bound to be incomplete. So let us ask our question differently: What are the basic changes in mathematical intuition or what questions are natural for us but could not be imagined at the beginning of the century?

There are some areas which have seen immense progress but where we do not find much change in mathematical intuition. For example I think that the sentence of Poincaré written at the turn of the century, "Analysis profits by geometric considerations, as it profits by the problems it is obliged to solve in order to satisfy the requirements of physics", adequately describes our contemporary understanding of analysis. Therefore I will not talk about the development of Analysis in this century. Rather, I will choose topics which in my opinion represent the basic shifts in mathematical perspective. Of course I can only present my personal views and different mathematicians will see the mathematical landscape in a completely different light.

Also, I prefer not to start from a discussion of particular mathematical achievements. Instead let us begin by considering the old question: "How is mathematics *possible*?" One of the possible interpretations of this question is, "How are we mathematicians able to perform our work?"

One of the main themes of 19th century mathematics was to "make mathematics rigorous." At the beginning of this century, therefore, the question, "How is mathematics *possible*?", might have been interpreted as the twofold directive:

1. set up a formalism adequate for mathematical reasoning and prove that such a formalism does not lead to contradictions, and
2. show that any question can be resolved.

The development of mathematics in the 20th century banished any hope for such a "naive" understanding. Godel has shown that we can never be sure that our framework, our chosen system of axioms does not lead to contradictions. Moreover we now know that any [sufficiently rich] system of axioms is incomplete. In other words if we are working in a framework of a sufficiently rich system of axioms then either our system leads to a contradiction or we meet statements about which we will never have anything to say. That is, we will neither be able to prove these statements, nor to disprove them, nor to show that we cannot either prove or disprove them. At first glance Godel seems to have signed the death sentence for mathematics. One would expect the unsolvable questions to jump at us in large numbers. If that were really the case, we would never be sure whether it makes sense to try seriously to solve difficult problems; surely, then, mathematics would come to a halt.

Fortunately, reality is very different. Aside from some very specific areas, we seem rarely to run into questions which we cannot settle and even in these areas we are sometimes able to prove that the questions we can't answer are "independent", that is, we know that we can neither prove nor disprove these statements. In view of this, we now give the old question, "How is mathematics possible?", a new interpretation: What is the mechanism which so often leads us to ask "meaningful" questions, i.e., questions which can be resolved?

I do not think that anyone has even an inkling of where to look for an answer to this. But I think that our ability to avoid the prognostication that might be suggested by Godel's theorem is related to the well-known but surprising observation that it is easier to solve a more general problem than a specific one. You see, there is a big difference between generalizations in mathematics and generalizations in social studies. In the case of social studies we pay for any generalization by being forced to accept an increasing number of counterexamples. In contrast, in mathematics where exceptions are not allowed, the existence of a sufficiently general statement to which we cannot find counterexamples is a strong indication that the statement is provable. [For example many people thought that the Fermat conjecture could neither be proved nor disproved nor shown to be undecidable. But immediately after Frey realized that the Fermat conjecture follows from the much more general Taniyama–Weil conjecture it became "clear" that Fermat's conjecture would be solved.]

We can also ask: "How is *mathematics* possible?" or, "Why doesn't mathematics split into a number of unrelated disciplines?" When one reads writings from the turn of the century one sees that the explosion of mathematics was seen to be the main problem which could destroy the unity of mathematics. Even then there was no mathematician who could follow all the developments; mathematics threatened to become a bunch of unrelated disciplines. Poincaré writes: "An attempt is made to cut it in pieces – to specialize. Too great a movement in this direction constitutes a serious obstacle to the progress of science." How could unity be preserved?

A choice of an answer to this question depends greatly on the answer to the first question: "How is mathematics *possible*?"

A "formal" interpretation of the first question" represents a very specific understanding of the structure of mathematics whereby logical structure takes on primary importance. This interpretation suggests Hilbert's one explanation for the unity – the main uniting force comes from the common structure: the logic of proofs.

On the other hand, Poincaré, for whom mathematics is characterized by the "economy of thought", writes that the unity of mathematics will be preserved by unexpected concurrencies as mathematics progresses.

We see now that both Hilbert and Poincaré are right – mathematics was able to preserve the unity during the multifaceted development of the 20th century and this unity is due both to the structural clarity and the immense number of unexpected connections between different areas of mathematics.

Actually the question, "How is mathematics *possible*?", was already asked by Kant who understood it as the question, "*How* is mathematics possible?" Kant saw the existence of mathematics as a proof for the existence of pure intuition. Mathematics for Kant was Euclidean. Such an understanding of mathematics does not correspond to everyday experience which teaches that some statements which are "intuitively clear" to one mathematician could be "counter-intuitive" to another. As Poincaré described beautifully in his article "Mathematical Discovery", an unexpected immediate illumination sometimes comes after a long and often seemingly unproductive period.

In other words mathematical intuition is not a natural phenomenon, is not given at birth, but develops throughout one's lifetime. I prefer to discuss the change in intuition of the mathematical community rather than follow the development of intuition of a particular mathematician [the topic of the article "Mathematical Discovery].

I think that the most drastic change in mathematical intuition came from the development of algebra. At the end of the previous century it was possible to subdivide mathematics into Algebra and Analysis, which contained Geometry, and these two areas were almost independent. At the end of this century we find ourselves in the position where the majority of achievements in Analysis and Geometry are, at least partially, based on the development of algebraic intuition. It is very characteristic that such a brilliant mathematician as Pontriagin dropped mathematics after the appearance of the post-war French school which was based on new algebraic intuition. This new understanding that the analysis of different algebraic structures is central for the development of mathematics found the most striking expression in the development of the category theory. I do not think that it would be possible to explain the basics of the category theory to any mathematician of the last century. The reason is that the theory of categories is "too simple". This theory, which originated in the forties, is based on a drastic shift of perspective: instead of studying the logic of the properties of mathematical objects the category theory studies the logic of relations. The category theory is perhaps the first serious extension of Aristotelian logic. In Aristotelian logic all the statements are "absolutely trivial" but in spite of this triviality Aristotelian logic is the backbone of all sciences. Analogously all the basic statements of the category theory are absolutely trivial but this logic of relations is the basis for a big chunk of modern mathematics. It is very significant that the first paper on the category theory was rejected by a first-rate mathematical journal for lack of content.

How does this new way of thinking change mathematical reality? It is impossible to describe the full picture while standing on one foot but I can give two applications of this new way of thinking. The first application is the possibility of constructing "ideal" objects which are completely defined in terms of their relations with the previously known objects. The second advantage coming from the category theory is the possibility of seeing familiar mathematical objects as "materializations" or, if you wish, shades of the more elaborate and structured objects. For example, much of the recent progress in representation theory is based on the understanding that, in a number of cases, functions are "materializations" of more elaborate algebraic-geometric objects.

The third topic I want to discuss is the change in structure of the interrelation between mathematics and physics. There were two different stages to this change.

In the first stage which started already at the beginning of this century, physicists realized that they needed mathematics not only as a tool to solve their problems, but also as a language to formulate laws of physics. Both the relativity theory and quantum mechanics rely on "modern" mathematics for the formulation of "physical" reality. There is no way to explain some of the most basic problems of contemporary physics to people who do not have an extensive mathematical background. But in this first stage, we still find a familiar structure to the relation between mathematics and physics when mathematics is used by physicists as a tool for the formulation and solution of their problems. The second stage, which started 20 years ago, brought a reversal of roles. In the last two decades of this century, we have had an increasing number of examples of applications of physics to mathematics. These applications are primarily in the form of conjectures which relate mathematical problems that were viewed by mathematicians as having nothing in common. How is this possible? In many physical theories there is a way to express physical quantities in terms of a functional integral. Since the functional integral does not have a rigorous definition such expressions do not have any meaning for mathematicians. Imagine now that the problem we consider depends on a parameter [say energy] and in the case when the energy is either very high or very low, there is a way to approximate the corresponding functional integrals by conventional mathematical expressions. These conventional expressions for the case of low and high energy are very different and we obtain two different rigorous expressions for the physical quantities – one from the analysis of the case when the energy is high and the other when it is low. From the point of view of physics both expressions are specializations of the original functional integral. Therefore "physical intuition" implies that these two different expressions coincide. On the other hand there is no obvious mathematical explanation for such a coincidence.

The existence of mathematical consequences of physical theories leads to the situation where mathematics plays a role of experimental physics for some branches of theoretical physics. It has become either impossible or too expensive to check the validity of some physical theories by experiment. Instead the validity of a physical theory is "confirmed" by the correctness of the mathematical predictions which can be deduced from this theory.

The last topic I want to discuss is the appearance of computer science. As a result of this development, mathematicians realized that it is not sufficient to ask whether a particular problem is solvable, but one should also inquire whether it can be solved in a reasonable amount of time. Computer scientists defined "reasonable" questions as such questions where you can check the correctness of an answer in a short [=polynomial] time. On the other hand one can consider a more restricted group of questions which can be solved in a short time. The basic problem of computer science is whether these two groups are really different, whether P≠NP. At first glance it is "clear" that P≠NP, that there are many ways to ask "reasonable" questions which are difficult to solve. But as we have already discussed, mathematical problems have a strong tendency to be solved in a relatively short time. Really, if a solution to a particular mathematical problem would take an exponentially long time we would never be able to solve such a problem. So either P=NP or we, mathematicians, are somehow able to choose very special "solvable" questions. Therefore we can restate the question, "Why are we mathematicians able to perform our work?", in a stronger form. We can ask: "What is the mechanism which leads us to ask questions which can be solved and can be solved in real time?"

David Kazhdan, Department of Mathematics, Harvard University, Cambridge, MA 02138, USA kazhdan@math.harvard.edu