

# Around two theorems and a lemma by Lucio Russo

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## Abstract

We describe two directions of study following early work of Lucio Russo. The first direction follows the famous Russo-Seymour-Welsh (RSW) theorem. We describe a RSW-type conjecture by the first author which, if true, would imply a coarse version of conformal invariance for critical planar percolation. The second direction is the study of "Russo's lemma" and "Russo's 0-1 law" for threshold behavior of Boolean function. We mention results by Friedgut, Bourgain, and Hatami, and present a conjecture by Jeff Kahn and the second author which may allow applications for finding critical probabilities.

## 1 Introduction

We did not meet Lucio Russo in person but his mathematical work greatly influenced our own and his wide horizons and interests in physics, mathematics, philosophy, and history greatly inspired us. We describe here two directions of study following early work of Russo. The first section follow the famous Russo-Seymour-Welsh theorem regarding critical planar percolation. The second section follows the basic "Russo's lemma" and deep "Russo's 0-1 law". In each direction we present one central conjecture.

## 2 Planar percolation

Consider  $1/2$ -Bernoulli percolation on a planar grid. Russo [24] and Seymour and Welsh [26] proved the *RSW theorem* comparing the probability of having an open crossing in a  $n \times cn$  rectangle to that of crossing a square. In particular implying that:

*the probability critical Bernoulli percolation crosses a long rectangle is bounded away from zero and is depending only on the aspect ratio.*

This fundamental fact is used and extended to a variety of models, involving rather clever proofs. But for now all proofs are using rotation symmetry. Vincent Tassion recently gave a couple of proofs for the RSW fact under various sets of weaker assumptions, key to solving several known problems [30]. Personally let us note that the RSW lemma was essential in controlling the influence of a fixed edge on the crossing event, allowing us to establish, jointly with Oded Schramm, noise sensitivity of critical percolation, see [2, 15].

What about a RSW type result for a more general planar graphs going beyond Euclidean lattices and tessellations?

In what follows we would like to suggest a conjectural extension of RSW theorem to general planar triangulations. The motivation comes from conformal uniformization, see [1].

## 2.1 A generalized RSW conjecture.

Tile the unit square with (possibly infinity number) of squares of varying sizes so that at most three squares meet at corners. That is, the dual graph is a triangulation. Color each square black or white with equal probability independently.

**Conjecture 2.1.** *There is a universal  $c > 0$ , so that the probability of a black left right crossing is bigger than  $c$ .*

At the moment we do not have a proof of the conjecture even when the squares are colored black with probability  $2/3$ . Behind the conjecture is a coarse version of conformal invariance. That is, the crossing probability is balanced if the tiles are of uniformly bounded distance to circles (rotation invariance), and the squares can be of different sizes, (dilation invariance). If true the same should hold for a tiling or a packing of a triangulation, with a set of shapes that are of bounded Hausdorff distance to circles.

If the answer to conjecture 2.1 is affirmative, this will imply the following: Let  $G$  be the 1-skeleton of a bounded degree triangulation of an open disk. Assume  $G$  is transient for the simple random walk, then  $1/2$ -Bernoulli site percolation on  $G$  admits infinitely many infinite cluster a.s. We do not know it even for any  $p$ -Bernoulli percolation with  $1 > p > 1/2$ . By [3] such triangulations result in a square tilings as in the conjecture. the proof in [3] is an analogous of the RSW phenomena for simple random walk on the triangulation. We speculate  $1/2$ -Bernoulli site percolation on  $G$  admits infinitely many infinite cluster a.s. *iff*  $G$  is transient.

How does the *influence* of a square in the tiling on the crossing probability at  $p = 1/2$  and its area related? Establishing a *high dimensional* version of the RSW lemma is a well

known very important open problem. Either at  $p = 1/2$  for two dimensional plaquettes in four dimension using duality, or in three dimension place independently 2 cells in the 3D square grid. Informally look at the critical  $p$  for full infinite surface and prove RSW for plaquettes in cubes. That is, if the probability of no open path from top to bottom in a  $n \times n \times n$  box with probability at least  $1/2$ , then no open path from top to bottom in a cube  $2n \times 2n \times n$ , with probability bounded away from 0 independently of  $n$ .

*A comment on large graphs and percolation.* In the category of planar graphs, in view of the (discrete) conformal uniformization, transience (equivalently *conformally hyperbolic*) is a natural notion of largeness. In the context of Cayley graphs *nonamenability* serves as a notion of large Cayley graphs. Thus the still open conjecture ([4]) that there is a non empty interval of  $p$ 's so that  $p$ - Bernoulli percolation admits infinitely many infinite clusters *iff* the group is nonamenable, shares some flavor with conjecture 2.1. As both suggest that a graph is large provided there is a phase with infinitely many infinite clusters.

### 3 Isoperimetric inequalities and Russo's 0-1 law

We endow the discrete cube  $\Omega_n = \{-1, 1\}^n$  with the probability product measure  $\mu_p$  where the probability for each bit to be 1 is  $p$ . A Boolean function  $f$  is a function from  $\Omega_n$  to  $\{-1, 1\}$ , and  $f$  is monotone if changing the value of a variable from -1 to 1 does not change the value of  $f$  from 1 to -1. The influence of the  $k$ th variable on  $f$  denoted by  $I_k^p(f)$  is the probability that changing the  $k$ th variable will change the value of  $f$ . The total influence is  $I^p(f) = \sum_{k=1}^n I_k^p(f)$ . We denote  $\mu_p(f) = \mu_p\{x : f(x) = 1\}$ , and write  $\text{Var}_p(f) = 4\mu_p(f)(1 - \mu_p(f))$ . (If  $p = 1/2$  we omit the superscript/subscript  $p$ .)

A basic result in extremal and probabilistic combinatorics going back to Harper (and others) is the isoperimetric inequality. For the measure  $\mu_p$  the isoperimetric relation takes the form (see, e.g. [19, 20]):

**Theorem 3.1.**

$$pI^p(f) \geq \mu_p(f) \log_p(1/\mu_p(f)).$$

If  $f$  is monotone then  $\mu_p(f)$  is a monotone function of  $p$ . Fixing a small  $\epsilon > 0$  the threshold interval of  $f$  is the interval  $[p, q]$  where  $\mu_p(f) = \epsilon$ ,  $\mu_q(f) = 1 - \epsilon$ . A fundamental lemma by Russo [25] and Margulis [22] asserts that for a monotone Boolean function  $f$ ,

$$d\mu_p(f)/dp = I^p(f).$$

The deep Russo’s 0-1 law [25] asserts informally that the threshold interval of a Boolean function is of size  $o(1)$  if all variables have  $o(1)$ -influence. In view of the Russo-Margulis lemma understanding the total influence is crucial for understanding the threshold window of a Boolean function. Sharp form of Russo 0-1 theorem and various related results were proved in the last two decades and Fourier methods played an important role in these developments. We mention especially the paper by Kahn, Kalai and Linial [18] and subsequent papers [8, 28, 14, 12], and books [15, 23]. To a large extent, this study is centered around the following problem.

**Problem:** Understand the structure of Boolean functions of  $n$  variables for which

$$I^p(f) \leq K \frac{1}{p} \mu_p(f) \log_p(1/\mu_p(f)).$$

We will quickly describe some main avenues of research and central results regarding this problem. For a more detailed recent survey see [20].

1. For the case where both  $p$  and  $\mu_p(f)$  are bounded away from zero and one (or even when  $\log(1/p)/\log n \rightarrow 0$ ) and  $K$  is bounded, Friedgut (1998) [12] proved that such functions are approximately “juntas,” namely, they are determined (with high probability) by their values on a fixed bounded set of variables. This result can be seen as a sharp form of Russo’s 0-1 law and it has a wide range of applications.
2. For the case where  $K$  is bounded,  $\mu_p(f)$  is bounded away from zero and one, but  $\log(1/p)/\log n$  is bounded away from zero, there are important theorems by Friedgut (1999) [13], Bourgain (1999) [7] (see below), and Hatami (2010) [17]. These results have important applications for proving sharp thresholds theorems. Hatami’s work is based on a mysterious while important notion of pseudojuntas.
3. The case where  $K$  is bounded, and  $\mu_p(f)$  is small is wide open. This case is important on its own and may have some applications for finding the critical probability, see Conjecture 3.3.
4. Cases where  $K = 1 + \epsilon$  are of different nature and are also of much interest. See, e.g. [10, 11].
5. There are few results regarding the case that  $K$  is unbounded and especially when  $K$  grows quicker than  $\log n$ . (One such result is by Bourgain and Kalai for functions with various forms of symmetry [9].) This is of great interest already when both  $p$  and  $\mu_p(f)$  are bounded away from 0 and 1.

We will mention now a theorem of Bourgain and a far-reaching related conjecture.

**Theorem 3.2** (Bourgain [7]). *There exists  $\epsilon > 0$  with the following property. For every  $C$  there is  $K(C)$ , such that if  $I^p(f) < pC$  then there exists a subset  $R$  of variables  $|R| \leq K(C)$  such that*

$$\mu_p(x : f(x) = 1 | x_i = 1, i \in S) > (1 + \epsilon)\mu_p(f).$$

**Conjecture 3.3** (Kahn and Kalai Conjecture 6.1 (a) from [19]). *There exists  $\epsilon > 0$  with the following property. For every  $C$  there is  $K(C)$ , such that if  $I^p(f) < pC\mu_p(f) \log(1/\mu_p(f))$  then there exist a subset  $R$  of variables  $|R| \leq K(C) \log(1/\mu_p(f))$  such that*

$$\mu_p(x : f(x) = 1 | x_i = 1, i \in S) > (1 + \epsilon)\mu_p(f).$$

Several attempts for stronger conjectures (such as Conjectures 6.1(b), 6.1 (c) from [19]) turned out to be incorrect. We conclude with another approach for understanding Boolean functions with small influence. The first step is the important Fourier-Walsh expansion. Every Boolean function  $f$  can be written as a square free polynomial  $f = \sum \hat{f}(S)x_S$ , where  $x_S = \prod\{x_i : i \in S\}$ . (The coefficients  $\hat{f}(S)$  are called the Fourier coefficients of  $f$ .) It is easy to verify that  $\sum \hat{f}^2(S) = 1$  and that  $\sum \hat{f}^2(S)|S| = I(f)$ . It follows that

**Proposition 3.4.** *For every  $\epsilon > 0$  a Boolean function  $f$  can be  $\epsilon \cdot \text{Var}(f)$ -approximated by the sign of a degree- $d$  polynomial where  $d = \frac{1}{\epsilon}I(f)$ .*

However, we note that Boolean functions described as signs of low degree polynomials may have large total influence. Our next step is to consider representation of Boolean functions via Boolean circuits. Circuits allow to build complicated Boolean functions from simple ones and they have crucial importance in computational complexity. Starting with  $n$  variables  $x_1, x_2, \dots, x_n$ , a *literal* is a variable  $x_i$  or its negation  $-x_i$ . Every Boolean function can be written as a formula in conjunctive normal form, namely as AND of ORs of literals. A *circuit* of depth  $d$  is defined inductively as follows. A circuit of depth zero is a literal. A circuit of depth one consists of an OR or AND *gate* applied to a set of literals, a circuit of depth  $k$  consists of an OR or AND gate applied to the outputs of circuits of depth  $k - 1$ . (We can assume that gates in the odd levels are all OR gates and that the gates of the even levels are all AND gates.) The size of a circuit is the number of gates. The famous **NP**  $\neq$  **P**-conjecture (in a slightly stronger form) asserts that the Boolean function described by the graph property of containing a Hamiltonian cycle, cannot be described by a polynomial-size circuit.

A theorem by Boppana [6] (the monotone case) and Hastad [16] (the general case) asserts that if  $f$  is described by a depth  $d$  size  $M$  Boolean circuit then  $I(f) \leq C(\log M)^{d-1}$ . We

conjecture that functions with low influence can be approximated by low-depth small size circuits. A function  $g$   $\delta$ -approximates a function  $f$  if  $|\mathbb{E}(f - g)^2| \leq \epsilon$ .

**Conjecture 3.5** (Benjamini, Kalai, and Schramm [2], 1999 (Slightly extended)). *For some absolute constant  $C$  the following holds. For every  $\epsilon > 0$  a Boolean function  $f$  can be  $\epsilon \cdot \text{Var}(f)$ -approximated by a circuit of depth  $d$  of size  $M$  where*

$$(\log M)^{Cd} \text{Var}(f) \leq I(f).$$

## 4 Conclusion

Congratulations, Lucio for your remarkable career and contributions and best wishes for the future. It is time for us to meet!

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